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Generalization of the Wittrick–Williams Formula for Counting Modes of Flexible Structures

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A recent (1971) result due to Wittrick–Williams relating the number of modes of a flexible structure to the zeros of a matrix function is generalized and an independent proof given. It is applied to Timoshenko models of a class of interconnected beamlike lattice structures and an interpretation of the zeros given as pinned modes of the structure.

I. Introduction

IN determining modes of continuum models of flexible structures, particularly when the mode density is high, it is often necessary to make sure that no modes have been missed. Here, a result due to Wittrick and Williams[1] (which is at once an algorithm and often referred to as such) may be used to determine the total number of modes in any given interval of frequency. In the words of the authors, it is an algorithm in the sense that it "enables one to calculate how many natural frequencies lie below any chosen frequency, without determining them and hence to converge to any required natural frequency to any specified accuracy." References 2–5 are illustrative of the actual use of this algorithm for determining modes of beamlike structures.

Our concern in this paper is in the formula itself rather than its use as a computational tool for finding modes of structures. Indeed, it can be regarded as a generalization of Foster's theorem for one-dimensional loss-less impedances, familiar in classical electrical circuit theory; see Ref. 6. Also, the proof given by the authors in Ref. 6 leans heavily on a classical theorem of Lord Rayleigh[7] and many arguments peculiar to mechanics of structures. Here we formulate the problem in the generality of an abstract wave equation in a Hilbert space and at the same time present an entirely independent mathematical proof. We show, in addition, how it can be applied to Timoshenko models of interconnected beamlike lattice structures.

II. Statement of Result and an Illustrative Example

We can state our main result quite simply. Let $\mathcal{H}$ denote a separable Hilbert space and M a self-adjoint positive definite linear bounded operator mapping $\mathcal{H}$ onto $\mathcal{H}$ with bounded inverse; this is then the mass-inertia matrix/operator. Let A denote a closed self-adjoint nonnegative linear operator with domain dense in $\mathcal{H}$ and range in $\mathcal{H}$. We assume that A has a compact resolvent and that zero is not in the spectrum. A is then the stiffness matrix/operator. Readers unfamiliar with these concepts, especially in the infinite-dimensional version, may consult Refs. 8 and 9 for a specific application. Then the main result is a characterization of the eigenvalues defined by

$$A\phi_k = \omega_k^2 M\phi_k$$

where the $\{\omega_k\}$ are structure mode frequencies or, simply, modes. Let B be a linear mapping of a Euclidean space E^n into $\mathcal{H}$, and let

$$H(\omega) = B^*(\omega^2 M + A)^{-1}B$$

Then under a controllability condition on B, we prove the following.

For each ω, $0 \leq \omega$, the number of modes (each counted to its multiplicity) in the interval $[0, \omega]$ is equal to the number of zeros (each counted to its multiplicity) of $H(\cdot)$ in the same interval, plus the number of negative eigenvalues (each counted to its multiplicity) of $H(\omega)$, at the point ω.

For a reasonably general class of structures including interconnected beamlike lattice trusses with a finite number of nodes (points of strain discontinuity) we can reduce $H(\omega)$ to the form where the operator inverse is replaced by a matrix inverse:

$$H(\omega) = B_u^*\left[-\omega^2 M_b + T(i\omega)\right]^{-1} B_u$$

where B_u is a rectangular (control) matrix, $T(i\omega)$ a real symmetric matrix, and M_b the compound matrix of mass/inertia matrices at the nodes. Moreover, the zeros of $H(\omega)$ can be shown to be modes of the structure with some or all of the nodes pinned (or clamped).

To illustrate the flavor of the result, let us consider a one-dimensional Timoshenko beam equation or pure beam torsion with one control actuator at one end and with the other end pinned, an elementary version of the general problem treated in Sec. IV. Thus, we have, in the usual notation for elastic constants with $\phi(t, s)$ denoting the torsion angle and $u(\cdot)$, the control

$$\rho\frac{\partial^2\phi}{\partial t^2} - G\frac{\partial^2\phi}{\partial s^2} = 0, \qquad 0 < t, 0 < s < L$$

$$\phi(t, 0) = 0$$

$$m\frac{\partial^2}{\partial t^2}\phi(t, L) + GI_\phi\frac{\partial^3}{\partial s\partial t^2}\phi(t, L) + u(t) = 0$$

Here we can readily calculate that

$$H(\omega) = [-m\omega^2 + T(i\omega)]^{-1}$$

where

$$T(i\omega) = I_\phi\sqrt{\rho G}\,\omega\cot(L\sqrt{\rho/G}\,\omega)$$

The zeros of $H(\omega)$ in this case are the poles of $T(i\omega)$, or the zeros of

$$\sin(L\sqrt{\rho/G}\,\omega)$$

which are recognized as the clamped–clamped modes. The number of modes less than or equal to ω is equal to the number of zeros or plus one depending on the sign of $H(\omega)$. Indeed, $H(\omega)$ being monotone nonincreasing between modes, we see that there is a mode between any two zeros: the modes and zeros alternate.

The main result and proof are in Sec. III. The proof is an improvement of an earlier version[6] to include the case of multiple eigenvalues. It is elementary except for invoking the theory of algebroidal functions characterizing the roots of polynomials with coefficients that are meromorphic functions and some rudimentary theory of self-adjoint operators with compact resolvents. Section IV is devoted to the application to beamlike lattice trusses modeled as anisotropic Timoshenko beams with a finite number of nodes. Conclusions are in Sec. V.

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III. Main Result

In this section we state and prove the main result. Let $\mathcal{H}$ be a separable Hilbert space. Let A denote a self-adjoint nonnegative definite operator with domain dense in $\mathcal{H}$ with compact resolvent and zero in the resolvent set ($\equiv$ no rigid-body modes, in the structures context). Let M denote a positive definite linear bounded operator on $\mathcal{H}$ onto $\mathcal{H}$ with bounded inverse.

Then (for example, see Ref. 8) A has a countable number of positive eigenvalues $\{\omega_k^2\}$ monotone increasing in k, and $\mathcal{H}$ is the sum of a corresponding sequence of M-orthonormal eigenfunction spaces M_k, each of at most finite dimension m_k

$$A\phi_{ik} = \omega_k^2 M\phi_{ik}, \qquad i = 1, \ldots, m_k \tag{1}$$

where

$$[M\phi_{ik}, \phi_{jk}] = \delta_j^i, \qquad i, j \leq m_k$$

For each k, the $\{\phi_{ik}\}$ span M_k. Also, $\{\phi_{ik}\}$ altogether span $\mathcal{H}$ and

$$[M\phi_{ij}, \phi_{k\ell}] = 0, \qquad j \neq \ell$$

We assume that

$$\sup_k m_k < \infty \tag{2}$$

(which is satisfied in the case of beamlike structures but not necessarily for plates).

Next let B denote a linear operator (the control operator) mapping the Euclidean space E^n into $\mathcal{H}$, such that B is 1:1 (B^*B is nonsingular). The crucial assumption, however, is that

$$B^* P_k x = 0 \tag{3}$$

implies $P_k x = 0$, where P_k is the projection operator corresponding to M_k. In particular, this implies that

$$b_{ik} = B^*\phi_{ik}, \qquad i = 1, \ldots, m_k$$

span a subspace of dimension m_k in E^n. It is necessary, therefore, that the controller dimension n must be large enough so that

$$n > \sup_k m_k \tag{4}$$

which, in particular, shows why assumption (2) is necessary. We shall see also that the condition (3) is necessary for the basic result (Theorem 1) to hold.

Let s denote a complex variable. Then,

$$(s^2 M + A)$$

has a bounded inverse for

$$s^2 + \omega_k^2 \neq 0$$

Hence,

$$\psi(s) = B^*(s^2 M + A)^{-1} B$$

defines an n by n matrix function of the complex variable s that is analytic in the finite part of the plane except for isolated poles at $s = \pm i\omega_k$. In fact, we have the expansion

$$\psi(s) = \sum_1^\infty \frac{Q_k}{s^2 + \omega_k^2} \tag{5}$$

where

$$Q_k = \sum_{i=1}^{m_k} b_{ik} b_{ik}^* \tag{6}$$

and

$$\sum_1^\infty \frac{\mathrm{tr} \cdot Q_k}{\omega_k^2} < \infty \tag{7}$$

Let

$$H(\omega) = \psi(i\omega)$$

$$= B^*(-\omega^2 M + A)^{-1} B = \sum_1^\infty \frac{Q_k}{\omega_k^2 - \omega^2} \tag{8}$$

This is a real symmetric matrix, defined for all $\omega \neq \omega_k$. Moreover a direct consequence of Eq. (3) is that for $\omega = \omega_k$,

$$[H'(\omega)u, u] > 0 \qquad \text{for} \qquad \|u\| \neq 0 \tag{9}$$

as is immediate from

$$H'(\omega) = 2\sum_1^\infty \frac{Q_k \omega}{\left(\omega_k^2 - \omega^2\right)^2}$$

and

$$[H(\omega)u, u] = 0$$

would imply that for every k

$$[Q_k u, u] = 0$$

or

$$0 = [b_{ik}, u] = [Bu, \phi_{ik}]$$

or

$$Bu = 0$$

implying $u = 0$. It is also immediate that the $\{b_{ik}\}$ span E^n since

$$[B_{ik}, u] = 0$$

for all i, k implies that $u = 0$.

For each $\omega \geq 0$, let $\sigma(\omega)$ denote the number of negative eigenvalues of $H(\omega)$, each eigenvalue being counted to its multiplicity. Let $z(\omega)$ denote the number of zero eigenvalues of $H(\cdot)$ in the interval $[0, \omega]$, each zero counted to its multiplicity. Let $p(\omega)$ denote the number of eigenvalues of A in the interval $[0, \omega]$, each counted to its multiplicity. Our main concern is, of course, the last function and our basic result, which generalizes the Wittrick–Williams formula, is the following.

Theorem 1.

$$p(\omega) = z(\omega) + \sigma(\omega), \qquad \omega \neq \omega_k, 0 < \omega \tag{10}$$

Proof. The proof is an extension of the one given in Ref. 6 to include the multiple eigenvalue case, and Ref. 6 may be consulted for more detail, as necessary.

Let $G(s, \lambda)$ denote the characteristic polynomial of $\psi(s)$,

$$G(s, \lambda) = |\lambda I - \psi(s)|_{\mathrm{Det}} = \lambda^n - p_{n-1}(s)\lambda^{n-1} + \cdots + p_0(s) \tag{11}$$

which is then a function of the two complex variables, s and λ. Since $\psi(s)$ is analytic except for isolated poles at $(\pm i\omega_k)$, the coefficient functions $p_k(s)$ are as well. The eigenvalues are algebroidal functions—an extension of the notion of algebraic functions[10]—and reference should be made to Refs. 11 and 12, for the analyticity properties used. Thus, except for a finite number of points in any bounded domain, not containing the poles $\{i\omega_k\}$, the eigenvalues are distinct and the distinct eigenvalues $\{\lambda_i(s)\}$ can be defined as holomorphic functions in the same domain, and also $\{h_i(s)\}$ as the corresponding eigenvectors. They are the branches of one and the same analytic function continuable across the suitably cut plane.[10] Since we are concerned mainly with the behavior on the imaginary axis, for simplicity of notation, let

$$d_k(\omega) = \lambda_k(i\omega)$$

$$e_k(\omega) = h_k(i\omega)$$

$$D(\omega) = p_0(i\omega) = \mathrm{Det}|H(\omega)|$$

At any nonexceptional point (this excludes $\{\omega_k\}$ in particular) we can differentiate

$$d_i(\omega) = [H(\omega)e_i(\omega), e_i(\omega)]$$

to obtain

$$\alpha_i'(\omega) = [H(\omega)e_i(\omega), e_i(\omega)] + [H(\omega)e_i'(\omega), e_i(\omega)] + [H(\omega)e_i(\omega), e_i'(\omega)]$$

where the last two terms

$$= d_i(\omega)([e_i'(\omega), e_i(\omega)] + [e_i(\omega), e_i'(\omega)])$$

$$= d_i(\omega)\frac{\mathrm{d}}{\mathrm{d}\omega}\|e_i(\omega)\|^2 = 0$$

Therefore, from Eq. (9) on the other hand,

$$d_i'(\omega) > 0$$

An immediate consequence of this is that any zero crossing of any eigenvalue $d_i(\omega)$ has to be from left to right on the ω axis: a negative value to a positive value.

Lemma 1. Given $\tilde{\omega}$, $\omega_k < \tilde{\omega} < \omega_{k+1}$. Suppose

$$D(\tilde{\omega}) \neq 0$$

Then

$$z(\omega) + \sigma(\omega) = z(\tilde{\omega}) + \sigma(\tilde{\omega}), \qquad \tilde{\omega} < \omega_k < \omega_{k+1}$$

Proof. Let $\tilde{\omega}$, $\omega_k < \tilde{\omega} < \omega_{k+1}$ be such that $H(\tilde{\omega})$ does not have zero for an eigenvalue. Now $D(\omega)$ is continuous in ω in any closed subinterval of (ω_k, ω_{k+1}). Hence, if there are no zeros in $\tilde{\omega} < \omega < \omega_{k+1}$, or

$$z(\omega) - z(\tilde{\omega}) = 0$$

it follows that

$$\sigma(\omega) - \sigma(\tilde{\omega}) = 0$$

since an eigenvalue, being continuous, cannot change sign without going through zero.

Next let us consider the case where $\tilde{\omega} + \Delta$ is the first zero in $\tilde{\omega} < \omega < \omega_{k+1}$. Let the multiplicity of the zero be m. As we have seen, zero crossing can occur when a negative eigenvalue changes to a positive eigenvalue. Hence, exactly three negative eigenvalues change sign. Hence, for $0 < \varepsilon$, ε small enough

$$\sigma(\tilde{\omega} + \Delta + \varepsilon) = \sigma(\tilde{\omega}) - m$$

and, of course,

$$z(\tilde{\omega} + \Delta + \varepsilon) = z(\tilde{\omega}) + m$$

Hence, again

$$\sigma(\omega) - \sigma(\tilde{\omega}) + z(\omega) - z(\tilde{\omega}) = 0, \qquad \tilde{\omega} < \omega < \omega_{k+1} \quad \text{QED}$$

Lemma 2. The number of negative eingenvalues $\sigma(\omega)$ has a discontinuity of the first kind (jump),

$$\sigma(\omega_k+) = \sigma(\omega_k-) + m_k - m_Z(\omega_k)$$

where m_k is the dimension of the range space of Q_k (= multiplicity of mode), and $m_Z(\omega_k)$ is the multiplicity of the zero, $m_Z(\omega_k) = 0$ if ω_k is not a zero.

Proof.

$$\left(s^2 + \omega_k^2\right)\psi(s)$$

is holomorphic in a domain containing $i\omega_k$. Since

$$\left(-\omega^2 + \omega_k^2\right)H(\omega) \to Q_k \qquad \text{as} \qquad \omega \to \omega_k$$

the corresponding eigenvectors (and eigenvalues) converge. Let us use the representation

$$Q_k = \sum_1^{m_k} \gamma_i h_i h_i^*, \qquad \gamma_i > 0; \qquad h_i^* h_j = \delta_j^i$$

We can renumber if necessary so that

$$\left(-\omega^2 + \omega_k^2\right)d_i(\omega) \to \gamma_i \qquad i = 1, \ldots, m_k$$

Also, $d_i(\omega)$ has the Laurent expansion

$$d_i(\omega) = \frac{\gamma_i}{\omega_k^2 - \omega^2} + R_i(\omega), \qquad i = 1, \ldots, m_k$$

where $R_i(\cdot)$ is continuous at $\omega = \omega_k$. Hence for $\omega^2 = \omega_k^2 - \delta^2$, we have

$$d_i(\omega) = \gamma_i/\delta^2 + R_i(\omega)$$

whereas for $\omega^2 = \omega_k^2 + \delta^2$, we have

$$d_i(\omega) = -\gamma_i/\delta^2 + R_i(\omega)$$

Hence, we see that for δ small enough, each eigenvalue $d_i(\omega)$, $i = 1, \ldots, m_k$ changes from a positive eigenvalue to a negative eigenvalue as ω increases from $\omega_k^2 - \delta^2$ to $\omega_k^2 + \delta^2$. The other eigenvalues of Q_k being zero, we have that for $m_{k+1} \leq i \leq n$, eigenvalue $d_i(\omega)$ is continuous. Let $m_Z(\omega_k)$ denote the multiplicity of the zeros. Then,

$$\sigma(\omega_k+) = \sigma(\omega_k-) + m_k - m_Z(\omega_k)$$

Proof of Theorem. The proof is by induction.

For $0 < \omega < \omega_1$, $H(\omega)$ is nonnegative definite and so is $(\mathrm{d}/\mathrm{d}\omega)H(\omega)$. But

$$[H(0)u, u] = \sum_1^{\infty} \frac{[Q_k u, u]}{\omega_k^2} = 0$$

implies $u = 0$ and, hence, $H(\omega)$ is nonsingular in $0 < \omega < \omega_k$ and hence the theorem holds in this range with $p(\cdot)$, $z(\cdot)$, and $\sigma(\cdot)$ all zero.

Suppose it holds for $\omega_N < \omega < \omega_{N+1}$. We shall show now that it holds for $\omega_{N+1} < \omega < \omega_{N+2}$. Thus, let

$$p(\omega) = z(\omega) + \sigma(\omega) \qquad \omega_N < \omega < \omega_{N+1}$$

where we note that

$$p(\omega) = \sum_1^N m_k$$

By Lemma 2,

$$\begin{aligned}\sigma(\omega_{N+1}+) &= \sigma(\omega_{N+1}-) + m_{N+1} - m_Z(\omega_{N+1}) \\ &= p(\omega_{N+1}-) - z(\omega_{N+1}-) + m_{N+1} - m_Z(\omega_{N+1}) \\ &= p(\omega_{N+1}+) - z(\omega_{N+1}+)\end{aligned}$$

or

$$p(\omega_{N+1}+) = z(\omega_{N+1}+) + \sigma(\omega_{N+1}+)$$

But by Lemma 1,

$$z(\omega) + \sigma(\omega) = z(\omega_{N+1}+) + \sigma(\omega_{N+1}+) \quad \text{for} \quad \omega_{N+1} < \omega < \omega_{N+2}$$

But

$$p(\omega_{N+1}+) = p(\omega) \qquad \text{for} \qquad \omega_{N+1} < \omega < \omega_{N+2}$$

Hence,

$$p(\omega) = z(\omega) + \sigma(\omega) \qquad \text{for} \qquad \omega_{N+1} < \omega < \omega_{N+2}$$

This completes the induction.

From the proof it is easy to see why condition (3) is necessary. For if for some mode ϕ_{ik}

$$A\phi_{ik} = \omega_k^2 M\phi_{ik}$$

we have

$$B^*\phi_{ik} = 0$$

then Q_k would have dimension $(m_k - 1)$ and hence Lemma 2 would not yield correct mode multiplicity. In control terms this means also that this mode ϕ_{ik} would not be stabilizable.

IV. Application to Interconnected Beamlike Lattice Structures

The basic result, Theorem 1, becomes computationally useful only if the infinite dimensional inverse

$$(s^2M + A)^{-1}$$

in the formula for $\psi(s)$ can be replaced equivalently by a finite-dimensional matrix.

We shall now show how this can be accomplished for a class of structures of practical importance—interconnected beamlike lattice trusses—for which reasonably faithful continuum models can be constructed, and for which in particular Eq. (3) can be proved to hold.

An example of the kind of structure envisaged is given by the NASA Langley Flight Research Center evolutionary model,[13] a continuum model for which is developed in Ref. 14. Not to complicate details unduly, we shall here consider a generic anisotropic Timoshenko beam model[15] with a finite number of nodes (discontinuities in the strain) where actuators and/or lumped masses may be located. Referring to Refs. 16 and 17 for more details as necessary, let s denote the axis variable, $0 < s < L$, with L the beam length. The displacement vector is defined by

$$f(s,t) = \begin{vmatrix} u(s,t) \\ v(s,t) \\ w(s,t) \\ \phi_1(s,t) \\ \phi_2(s,t) \\ \phi_3(s,t) \end{vmatrix} \qquad 0 < s < L; \qquad 0 < t \tag{12}$$

where u, v, and w are the linear displacements and ϕ_1, ϕ_2, and ϕ_3 the (torsion) rotation angles. Let s_i, $i = 1, \ldots, N$, denote the nodes, $s_1 = 0$, $S_N = L$. The motion is described by the Timoshenko equations valid between nodes and the boundary conditions at the nodes.

The state space is a Hilbert space denoted $\mathcal{H}$ being the cross-product space

$$\mathcal{H} = L_2[0, L] \times E^{6N}$$

where $L_2[0, L]$ is the L_2 space of $6N$ by 1 function $f(\cdot)$ such that

$$\int_0^L \|f(s)\|^2 \, ds < \infty$$

We denote elements of $\mathcal{H}$ by

$$x = \begin{vmatrix} f \\ b \end{vmatrix}, \qquad f \in L_2[0, L]; \qquad b \in E^{6N}$$

The operator A is now defined on a dense domain in $\mathcal{H}$ (see Refs. 16 and 17 for more precise details)

$$Ax = \begin{vmatrix} g \\ A_b f \end{vmatrix} \tag{13}$$

where

$$g(s) = -A_2 f''(s) + A_1 f'(s) + A_3 f(s), \qquad s_i < s < s_{i+1}$$
$$i = 1, \ldots, N-1 \tag{14}$$

where

$$A_2 = \begin{vmatrix} C_1 & 0 \\ 0 & C_3 \end{vmatrix}$$

$$C_1 = \begin{vmatrix} c_{11} & c_{14} & c_{15} \\ c_{14} & c_{44} & c_{45} \\ c_{15} & c_{45} & c_{55} \end{vmatrix}, \qquad C_3 = \begin{vmatrix} c_{66} & c_{36} & c_{26} \\ c_{36} & c_{33} & c_{32} \\ c_{26} & c_{32} & c_{22} \end{vmatrix}$$

where C_1 and C_3 are nonsingular, symmetric, and positive definite matrices,

$$A_1 = \begin{vmatrix} 0 & C_2 \\ -C_2^* & 0 \end{vmatrix}$$

$$A_3 = \mathrm{Diag}(0, 0, 0, 0, c_{55}, c_{44})$$

and, finally, $A_b f$ is defined by

$$A_b f = \begin{vmatrix} -L_1 f(0+) - A_2 f'(0+) \\ A_2(f'(s_2-) - f'(s_2+)) \\ \vdots \\ A_2(f'(s_i-) - f'(s_i+)) \\ \vdots \\ L_1 f(L-) - A_2 f'(L-) \end{vmatrix} \tag{15}$$

where

$$L_1 = \begin{vmatrix} 0 & -C_2 \\ 0 & 0 \end{vmatrix}$$

Let U denote the vector of control inputs

$$U = \begin{vmatrix} u_1 \\ \vdots \\ u_n \end{vmatrix}$$

where n is the number of controls. Let B_u show how the controls are distributed among the nodes $\{S_i\}$ so that B_u is $6N$ by n. Let B denote the control operator defined by

$$BU = x; \qquad x = \begin{vmatrix} 0 \\ B_u U \end{vmatrix} \tag{16}$$

We may as well (and do) assume that

$$B_u U = 0$$

implies $U = 0$, and, hence, B is 1:1 so that

$$BU = 0$$

implies $U = 0$. Finally, let M_0 denote the structure mass-inertia matrix and M_b the nodal mass-inertia matrix so that

$$Mx = \begin{vmatrix} M_0 f \\ M_b b \end{vmatrix}$$

$$M_0 f = g; \qquad g(s) = M_0 f(s)$$

and M_0 and M_b are nonsingular.

Then the Timoshenko equations can be given the abstract form

$$M\ddot{x}(t) + Ax(t) + BU(t) = 0 \tag{17}$$

where $U(\cdot)$ is the control input and the dot denotes derivative with respect to time.

The structure modes are defined by

$$A\phi = \omega^2 M\phi, \qquad \phi \neq 0 \tag{18}$$

where ω is the mode frequency, and ϕ the mode shape

$$\phi = \begin{vmatrix} f \\ b \end{vmatrix} \tag{19}$$

where

$$b = \begin{vmatrix} f(s_1) \\ \vdots \\ f(s_N) \end{vmatrix}$$

The controllability assumption is that

$$B^*\phi = B_u^* b \neq 0$$

This can be assured in various ways; see Refs. 16 and 18. We do, however, need to prove Eq. (3). For this (and other purposes), let us expand on (Eq. 18). Thus, we have

$$\omega^2 M_0 f(s) = -A_2 f''(s) + A_1 f'(s) + A_3 f(s), \qquad s_i < s < s_{i+1} \tag{20}$$

$$A_b f = M_b b \tag{21}$$

We first solve Eq. (19). For this purpose, for any complex number p, let

$$\mathcal{A}(p) = \begin{vmatrix} 0 & I \\ -A_2^{-1}\left(A_3 + p^2 M_0\right) & A_2^{-1} A_1 \end{vmatrix}_{6\times 6} \tag{22}$$

and let us generalize Eq. (19) to

$$-p^2 M_0 f(s) = -A_2 f''(s) + A_1 f'(s) + A_3 f(s)$$

$$s_i \leq s < s_{i+1} \tag{23}$$

where $f(\cdot)$ is continuous and $f'(\cdot)$ has only jump discontinuity at most $s = s_i$. Using Eq. (22) we then have

$$\begin{vmatrix} f(s) \\ f'(s) \end{vmatrix} = e^{\mathcal{A}(p)(s-s_i)} \begin{vmatrix} f(s_i) \\ f'(s_i+) \end{vmatrix}, \qquad s_i \leq s < s_{i+1}$$

and, in particular, we see that $f(\cdot)$ is completely specified by specifying the $6N$ by 1 vector:

$$c = \begin{vmatrix} f(0+) \\ f'(0+) \\ f'(s_2+) \\ \vdots \\ f'(s_{N-1}+) \end{vmatrix} \tag{24}$$

Thus, we can write

$$f = \mathcal{L}(p)c$$

and, in particular, we can express

$$b = \begin{vmatrix} f(s_1) \\ \vdots \\ f(s_N) \end{vmatrix} = T(p)c$$

$$A_b f = L(p)c$$

where $J(\cdot)$ and $L(\cdot)$ are (matrix) entire functions of p.

The modes are then determined from

$$A_b f = -p^2 M_b b$$

or, in terms of c,

$$L(p)c = -p^2 M_b J(p)c$$

and since c cannot be zero,

$$\left|L(p) + p^2 M_b J(p)\right|_{\text{Det}} = 0 \tag{25}$$

In particular, we see that the multiplicity cannot exceed $6N$, thus establishing Eq. (3).

We can now proceed to calculate

$$\psi(p) = B^*(p^2 M + A)^{-1} B \tag{26}$$

Letting

$$(p^2 M + A)^{-1} BU = x = \begin{vmatrix} f \\ b \end{vmatrix}$$

or

$$\begin{vmatrix} 0 \\ B_u U \end{vmatrix} = (p^2 M + A) \begin{vmatrix} f \\ b \end{vmatrix}$$

We see that with c as before in Eq. (24)

$$f = \mathcal{L}(p)c$$

$$\left[L(p) + p^2 M_b J(p)\right]c = B_u U$$

$$B^* x = B_u^* J(p)c$$

hence,

$$\psi(p) = B_u^*\left(J(p)\left(L(p) + p^2 M_b J(p)\right)^{-1}\right)B_u \tag{27}$$

We have thus achieved the result sought, reducing the operator inverse to a matrix inverse.

Let us proceed next to explore further the nature of the zeros of $\psi(i\omega)$. Now $J(p)$ being an entire function has at most a countable number of zeros and, hence, $J(p)^{-1}$ is a meromorphic function. Hence, we can write

$$\psi(p) = B_u^*\left(p^2 M_b + T(p)\right)^{-1} B_u \tag{28}$$

where

$$T(p) = L(p) J(p)^{-1}$$

For $p = i\omega$, we note that $\mathcal{A}(p)$ is real and, hence, $L(i\omega)$ and $J(i\omega)$ are real valued, therefore so is $T(i\omega)$, and with $\psi(i\omega)$ being symmetric, so is $T(i\omega)$. The simplest case to consider is to take

$$B_u = \text{identity}$$

(and this, in fact, is the case considered by Wittrick–Williams[1]). Then we have

$$\psi(i\omega) = \left(-\omega^2 M_b + T(i\omega)\right)^{-1}$$

where the matrix

$$-\omega^2 M_b + T(i\omega)$$

is called the condensed dynamic matrix in Ref. 1. The simplification now is that the negative eigenvalues of $\psi(i\omega)$ are the same as those of the condensed dynamic matrix and, further, the zeros of $\psi(i\omega)$ are the poles of $T(i\omega)$ and, in turn, the poles of $T(i\omega)$ are the zeros of $J(i\omega)$. Let us examine the zeros of $J(i\omega)$. We have

$$J(p)c = 0, \qquad c \neq 0$$

Letting

$$f = \mathcal{L}(p)c$$

we see that

$$b = \begin{vmatrix} f(s_1) \\ \vdots \\ f(s_N) \end{vmatrix} = 0$$

Hence, we have

$$\omega^2 M_0 f(s) = -A_2 f''(s) + A_1 f'(s) + A_0 f(s), \qquad s_i < s < s_{i+1}$$

$$f(s_i) = 0, \qquad i = 1, \ldots, N$$

But these are recognized as the modes of the structure with all boundaries pinned. Moreover, the number of negative eigenvalues of $\psi(i\omega)$ are the same as those of the condensed dynamic matrix. Therefore, the number of modes less than or equal to ω is equal to the number of pinned modes in the same range, plus the number of negative eigenvalues of $[-m\omega^2 + T(i\omega)]$.

Getting back now to the general case, we see that the zeros of $\psi(i\omega)$ are the modes of the structure such that a certain linear transformation (namely, B_u^*) of the boundary values is zero. In the case where the sensors are collocated with the actuators, this would mean that motion in the sensed components is arrested.

At another extreme, if B_u is one dimensional (only one control), then $\psi(i\omega)$ is one dimensional. In this case the structure modes are the pinned modes and the pinned modes must alternate, as in the elementary explicit example in Sec. II.

For calculation of the condensed dynamic matrix for interconnected multibeam structures see Ref. 18 and for Bernoulli-beam models see Refs. 4 and 9. Also, as noted in Ref. 1, the number of negative eigenvalues of a symmetric matrix may be found by Sturm's algorithm that relates it to the number of sign changes in the diagonal entries.

V. Conclusions

A result due to Wittrick–Williams[1] relating the number of modes of a flexible structure to the number of zeros of a matrix meromorphic function has been generalized in scope and a totally independent proof given. It is applied to Timoshenko beam models of a class of interconnected beamlike lattice structures where it is shown, in particular, that the zeros are in fact the modes of the structure when some or all of the nodes are pinned.

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